The Weil Conjectures and Étale Cohomology: A Hand-Waving Introduction

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May 5, 2009

1 A Weil-gue recollection

Recall (from way back in lecture 15) the Weil Conjectures, a series of now-proved theorems about the zeta function of a variety over a finite field.

Let X be a nonsingular, d-dimensional projective variety over the finite field $\mathbb{F}_q$ of q elements. Let N_k be the number of points on X over the field of q^k elements. The *zeta function* of X is defined as

$$\zeta(X,s) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k}(q^{-s})^k\right).$$

Often we will want to make the substitution $t = q^{-s}$. When we wish to consider the zeta function of X as a function of t, the definition becomes

$$Z(X,t) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} t^k\right).$$

With this set-up, we can now state two of the Weil Conjectures:

- *Rationality*: $Z(X,t)$ is a rational function of t.

  More specifically, $Z(X,t)$ has the following form:

  $$Z(X,t) = \frac{P_1(t)P_3(t)\dots P_{2d-1}(t)}{P_0(t)P_2(t)\dots P_{2d}(t)},$$

  where each P_i is a polynomial with integer coefficients, $P_0(t) = 1 - t$, $P_{2d}(t) = 1 - q^d t$, and for $1 \le i \le 2d-1$,

  $$P_i(t) = \prod_{j=1}^{\beta_i} (1 - \alpha_{i,j} t),$$

  where the $\alpha_{i,j}$ are algebraic integers, and $\beta_i \in \mathbb{N}$ for all i and j.

- *Riemann Hypothesis*: $|\alpha_{i,j}| = q^{i/2}$ for all i and j.

The other two assertions say that the zeta function satisfies a functional equation, just like all good zeta functions should, and that the β_i are the topological Betti numbers.

What is so astounding about the Weil Conjectures is that they provide a link between the discrete (the number of points on a variety over a finite field) and the continuous (topological notions such as Betti numbers).

One of the great triumphs of algebraic geometry (or at least for those with an arithmetic bent) of the 20th century was proving these conjectures. In the rest of this talk, I hope to illustrate how they were proved by way of constructing a suitable *cohomology theory*. Since "cohomology" is one of those irritating words in maths which is casually slipped into conversation without ever explaining what it means, I'll also attempt to give a crash course in (co)homology.

2 Rationality via Lefschetz

Although rationality was proved first using p-adic methods by Dwork in 1959, it was Grothendieck in 1964 who discovered the correct cohomology theory which proved everything except the Riemann hypothesis (which was proved by Deligne in 1973). This section will attempt to give an idea of why such a cohomology theory leads to rationality as an immediate corollary.

To start with, we state an elemental result whose proof can be found in, for example, Ireland & Rosen's *A Classical Introduction to Modern Number Theory*, page 155, Proposition 11.1.1.

Lemma. The zeta function is rational if and only if there exist complex numbers α_i and β_j such that

$$N_k = \sum_j \beta_j^k - \sum_i \alpha_i^k.$$

So it remains to find such α_i and β_j.

What Weil himself noticed was that it was conceivable that such an expression might be obtained from a formula from topology attributed to Lefschetz, called the Lefschetz Fixed-Point Formula.

Theorem. Lefschetz Fixed-Point Formula

Let Y be a topological space and $f : Y \longrightarrow Y$ a continuous mapping from Y to itself. Let Λ_f denote the number of fixed points of f, i.e. points $y \in Y$ such that $f(y) = y$, counted with appropriate multiplicities. Then

$$\Lambda_f = \sum_i (-1)^i \operatorname{Tr}(f|H^i(Y, \mathbb{Q})).$$

What this theorem says is that the fixed points can be obtained from the trace of the induced linear map $f|H^i(Y, \mathbb{Q})$ acting on the cohomology $\mathbb{Q}$-vector spaces $H^i(Y, \mathbb{Q})$ for each i.

Now Weil's insight was to realise that the points $X(\mathbb{F}_{q^k})$ on a variety X over $\mathbb{F}_{q^k}$ are precisely the fixed points of the q^k-th power Frobenius map. That is, let $\bar{X}$ denote X considered over an algebraic closure of $\mathbb{F}_{q^k}$, and let $\phi_{q^k} : \bar{X} \longrightarrow \bar{X}$ be the Frobenius map which takes each coordinate of $\bar{X}$ to its q^k-th power. Then the fixed points of ϕ_{q^k} are precisely the points of $X(\mathbb{F}_{q^k})$.

So, suppose that X had some sort of "nice" topology on it, or some suitably complicated mathematical structure comparable to topology, and that we could define some sort of cohomology theory on X.[1] Then, hopefully, we could use the Lefschetz fixed point theorem to show that

$$N_k = \#X(\mathbb{F}_{q^k}) = \sum_i (-1)^i \mathrm{Tr}(\phi_{q^k} | H^i(X, \mathbb{Q}_\ell)).$$

Now if you're willing to believe (as I am) that each $\mathrm{Tr}(\phi_{q^k}|H^i(X,\mathbb{Q}_\ell))$ is expressible as the k-th power of a complex number, then we are in the situation of the lemma, and thus the zeta function is rational.

So all that remains(!) is to discover the correct cohomology theory, called a *Weil cohomology theory*. It is no surprise, of course, that such a cohomology theory comes from étale cohomology.

3 But what is cohomology?

This section will attempt to give the briefest of introductions to what is meant by (co)homology.

In a nutshell, a (co)homology theory is a way of assigning a sequence of modules to a given mathematical structure X in order to encode information about X. The archetypal example comes from algebraic topology: the case when X is a topological space, and the assigned sequence of modules are all $\mathbb{Z}$-modules, i.e. abelian groups. This is the so-called *singular homology*.

Homology. For any homology theory, the sequence of modules $C_0, C_1, C_2, \ldots$ associated to X is called a *chain complex*, denoted $C_\bullet(X)$. In order for $C_\bullet(X)$ to be a chain complex, we must also have maps $\partial_n : C_n \longrightarrow C_{n-1}$ satisfying $\partial_n \partial_{n+1} = 0$ for all n. A chain complex is often written as a diagram

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 = 0.$$

The *homology groups* $H_n(X)$ are defined to be the quotient modules

$$H_n(X) = \ker \partial_n \,/\, \mathrm{im}\, \partial_{n+1}.$$

Often the modules $C_0, C_1, C_2, \ldots$ are free modules over some ring, generated by some elements, and so we could construct them as free modules over *any* ring R. The ring R is often called the *coefficient ring*. To distinguish between homology groups with different coefficient rings, we write them as $H_i(X, R)$.

Cohomology. For a cohomology theory, the sequence of modules $C^0, C^1, \ldots$ associated to X is called a *cochain complex*, denoted $C^\bullet(X)$. In order for $C^\bullet(X)$ to be a cochain complex, we must also have maps $d^n : C^n \longrightarrow C^{n+1}$ satisfying $d^{n+1}d^n = 0$ for all n. A cochain complex is often written as a diagram

$$\ldots \xleftarrow{d^{n+1}} C^n \xleftarrow{d^n} C^{n-1} \xleftarrow{d^{n-1}} \ldots \xleftarrow{d^1} C^1 \xleftarrow{d^0} C^0 = 0.$$

The *cohomology groups* $H^n(X)$ are defined to be the quotient modules

$$H^n(X) = \ker d^{n+1} \,/\, \mathrm{im}\, d^n.$$

[1] It turns out that the "correct" cohomology groups are vector spaces over the ℓ-adic numbers, $\mathbb{Q}_\ell$, rather than over $\mathbb{Q}$. Don't ask me why.

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Functoriality. The operators $H_n(*)$ and $H^n(*)$ turn out to be functors (covariant and contravariant respectively), so that a morphism $X \longrightarrow Y$ induces a morphism on the homology groups $H_n(X) \longrightarrow H_n(Y)$ or on the cohomology groups $H^n(Y) \longrightarrow H^n(X)$.

Now if we look back at the Lefschetz fixed-point formula, the terms involved are far less mysterious. The function $f : X \longrightarrow X$ is a continuous map, so for each i, f induces a $\mathbb{Q}$-linear map $f|H^i(X, \mathbb{Q})$ between the vector space $H^i(X, \mathbb{Q})$ and itself, thus it has a trace.

4 Étale Cohomology and ℓ-adic Cohomology

Now that we know what we are looking for, here is a summary of the key points in the search for the "correct" cohomology theory which proves the Weil conjectures.

No Nice Topology! The main problem was that for X a variety over a finite field, no one could come up with a topology on X which was nice enough to use existing cohomology theories.

Enter the Étale Category. Grothendieck attacked this problem by replacing the notion of "topology", which can be thought of as the category of open sets on X, with the étale category on X, $\acute{\mathrm{E}}\mathrm{t}(X)$.

Étale Cohomology. With this generalisation, it is possible to construct a cohomology theory, called *étale cohomology*, which has certain "nice" properties when the coefficient ring is $\mathbb{Z}/n\mathbb{Z}$ for n coprime to p, the characteristic of the finite field.

ℓ-adic Cohomology. However, in order to find "the" cohomology theory, one requires to consider the ℓ-adic cohomology, constructed from étale cohomology:

Let ℓ be a prime not equal to p. For each étale cohomology group $H^i(X, \mathbb{Z}/\ell^k\mathbb{Z})$, define the ℓ-adic cohomology group $H^i(X, \mathbb{Z}_\ell)$ to be the inverse limit:

$$H^i(X, \mathbb{Z}_\ell) = \lim_{\leftarrow} H^i(X, \mathbb{Z}/\ell^k\mathbb{Z}).$$

Finally, we eventually reach the correct cohomology groups that will allow us to use the Lefschetz fixed-point formula, and hence prove three of the four Weil conjectures:

$$H^i(X, \mathbb{Q}_\ell) = H^i(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell.$$

Exercise: Now prove the Riemann hypothesis.